COHERENT BEAM INSTABILITIES

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1 Introduction

The motion of a single particle in a storage ring is well determined by the external guide fields (dipole and quadrupole magnets, RF system, etc.), by the initial conditions and the synchrotron radiation. The many particles in a high intensity beam represent a sizeable charge and current which act as a source of electromagnetic fields (self-fields). These fields are modified through the boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a *frequency shift* (change of the betatron or synchrotron frequency), to an increase of a small disturbance of the beam, i.e. an *instability* or a *change of the particle distribution*, e.g. bunch lengthening. These phenomena are usually called *collective effects* since a collective or a coherent action of the many particles in the beam is involved.

As an example we consider bunches in a storage ring going through a cavity. Each bunch induces electromagnetic fields in this cavity which oscillate and slowly decay away. The next bunch, or the same bunch on the next turn, might find some field left and will be influenced by it. Such a case is illustrated in Fig. 1. The phase of the field seen in the next turn can be such that a small initial synchrotron oscillation of the bunch is increased. In each turn the oscillation is amplified, resulting in an exponentially growing instability.

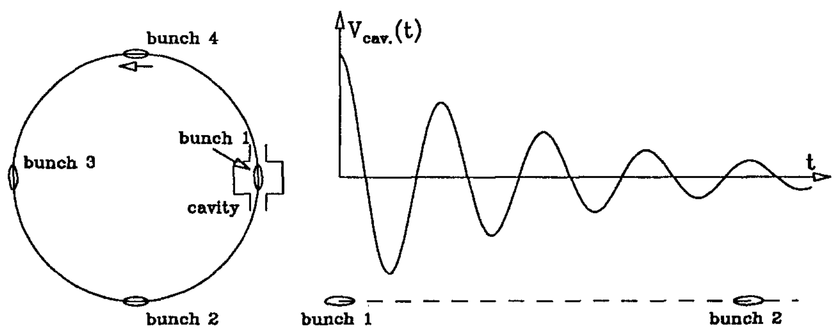


Figure 1: Field acting from one bunch to the next one

In many cases the fields created by the beam are small compared to the guide fields and their effects can be treated as a *perturbation*. This is done in three steps:

a) We calculate the stationary particle distribution which is given by the guide field, initial condition and synchrotron radiation.

b) We consider small disturbances of the bunch from the stationary distribution (different modes of betatron and synchrotron oscillations). We calculate the fields due to such a disturbance taking into account the boundary condition imposed by the beam surroundings (impedance).

c) We calculate the effect of these fields and investigate if the initial disturbance is increased *(instability)* or decreased *(damping)* or if the frequency of the mode of oscillation is changed *(frequency shift)*.

For the case of small self-fields considered here the particle distribution in the bunch is given by external conditions (machine parameter, initial condition, synchrotron radiation) and is usually Gaussian for electron machines. As disturbances of the stationary distribution we consider some modes of oscillation which are orthogonal to each other such that the stability of each mode can be treated independently.

For strong self-fields, however, the particle distribution is modified and the modes of oscillation are influenced such that they are no longer independent. A self-consistent solution has to be found in this case which is much more difficult. It is usually attempted only for the case of bunch lengthening where the longitudinal particle distribution in the presence of an impedance is to be obtained. The beam position monitors and the transverse impedance can usually not "resolve" the transverse particle distribution. It is therefore of importance for beam instabilities only in exceptional cases.

We distinguish between *single* and *multi-traversal* collective effects. For the first kind no memory of the induced field over one revolution or over the time distance between the passage of adjacent bunches is assumed. An example of a single-traversal effect is bunch lengthening. For multi-traversal effects the impedance has to have a memory such that one bunch can influence the next one or itself after one revolution. This is provided by cavity-like objects with a relatively large quality factor Q.

Finally, we distinguish between *longitudinal* and *transverse* effects. In the first case a longitudinal impedance influences the synchrotron oscillation such that its amplitudes grows or decays or changes its frequency. The transverse impedance has a corresponding effect on the horizontal or vertical betatron oscillations.

2 Impedance and wake potential of a resonator

2.1 Cavity resonance

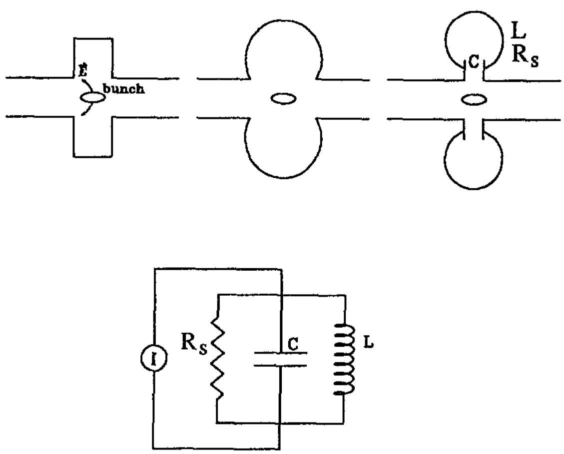


Figure 2: RLC circuit equivalent to a cavity resonance

Impedances and wake potentials have been treated extensively in an earlier lecture. As an introduction to the interaction between the beam and a cavity resonance we recapitulate here briefly some of the essential properties.

Cavities are the most important objects which can cause coupled-bunch mode instabilities since the induced fields oscillate for a relatively long time and provide a memory over one revolution. Such a cavity can be of a form which resembles an RCL circuit, Fig. 2, and can, in good approximation, be treated as such. The RCL circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these three parameters cannot easily be separated. For this reason one uses some other parameters which can easily be measured directly: The *resonance frequency* ω_r, the *quality factor* Q and the *damping rate* α:

$$\omega_r = \frac{1}{\sqrt{LC}}\,, \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C\omega_r\,, \quad \alpha = \frac{\omega_r}{2Q}. \tag{1}$$

If this circuit is driven by a current I we have the voltages across each element

$$V_R = I_R R_s\,,\; V_C = \frac{1}{C}\int I_C dt\,,\; V_L = L\frac{dI_L}{dt} \tag{2}$$

and the relations between the voltages and currents

$$V_R = V_C = V_L = V\,, \quad I_R + I_C + I_L = I. \tag{3}$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}. \tag{4}$$

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ we get the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}. \tag{5}$$

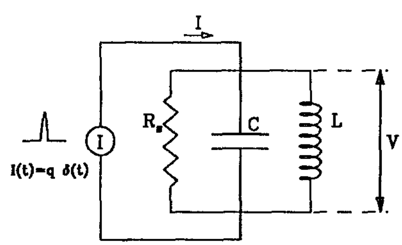
The solution of the homogeneous equation represents a damped oscillation

$$V(t) = \hat{V}e^{-\alpha t}\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}} + \phi\right) \tag{6}$$

or

$$V(t) = e^{-\alpha t}\left(A\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\,t\right) + B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\,t\right)\right). \tag{7}$$

2.2 Wake potential



$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$Q = R_s\sqrt{\frac{C}{L}}$$

$$\alpha = \frac{\omega_r}{2Q}$$

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

Figure 3: Equivalent RLC circuit driven by a delta pulse

We now calculate the response of the RCL-circuit (representing a cavity) to a delta function pulse (very short bunch), Fig. 3,

$$I(t) = q\delta(t) \tag{8}$$

The charge q will charge up the capacity to a voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q. \tag{9}$$

The energy stored in the capacitor is equal to the energy lost by the charge

$$U = \frac{q^2}{C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2, \tag{10}$$

where we introduced the *parasitic mode loss factor*

$$k_{pm} = \frac{\omega_r R_s}{2Q}. \tag{11}$$

The charged capacitor C will now discharge first through the resistor R_s and then also through the inductance L

$$\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = \frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2} q = \frac{2\omega_r k_{pm}}{Q} q. \tag{12}$$

The resonance circuit has now the initial conditions

$$V(0^+) = 2k_{pm} q \text{ and } \dot{V}(0^+) = \frac{2\omega_r k_{pm}}{Q} q\,. \tag{13}$$

Taking the solution of the homogeneous differential equation and its derivative

$$V(t) = \mathrm{e}^{-\alpha t}\left(\mathrm{A}\cos\left(\omega_r\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_r\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right) \tag{14}$$

$$\begin{aligned}\dot{V}(t) = \mathrm{e}^{-\alpha t} \quad & \left(\left(-A\alpha + B\omega_r\sqrt{1-\frac{1}{4Q^2}}\right)\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\,t\right)\right.\\ & \left.-\left(B\alpha + A\omega_r\sqrt{1-\frac{1}{4Q^2}}\right)\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\,t\right)\right)\end{aligned}$$

and satisfying the above initial conditions with

$$A = 2k_{pm} q \quad \text{and} \quad -A\alpha + B\omega_r\sqrt{1-\frac{1}{4Q^2}} = -\frac{2\omega_r k_{pm}}{Q} q \tag{15}$$

gives for the voltage in a resonator circuit excited at the time $t = 0$ by a δ-pulse $I(t) = q\delta t$

$$V(t) = 2qk_{pm}\mathrm{e}^{-\alpha t}\left(\cos\left(\omega_r\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_r\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right). \tag{16}$$

This voltage is induced by a charge q going through the cavity at the time $t = 0$. A second point charge q' going through the cavity will gain or lose the energy $U = q'V(t)$. This energy gain/loss per unit source and unit probe charge is called the *wake potential* of a point charge or also the *Green function* $G(t)$. For our resonator (cavity resonance) we have

$$G(t) = 2k_{pm}\mathrm{e}^{-\alpha t}\left(\cos\left(\omega_r\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_r\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right) \tag{17}$$

which for a large quality factor $Q \gg 1$ simplifies to

$$G(t) \approx 2k_{pm}\mathrm{e}^{-\alpha t}\cos(\omega_r \mathrm{t})\,. \tag{18}$$

2.3 Impedance

We assume now a *harmonic* excitation of the circuit with a current $I = \hat{I}\cos(\omega t)$, Fig. 4.

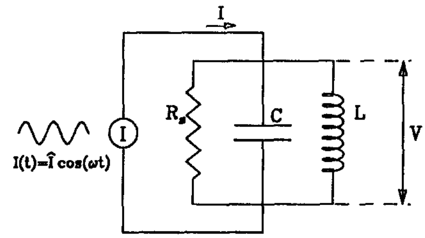


Figure 4: Equivalent RLC driven by a harmonic excitation

The differential equation of the harmonic excitation is

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t). \tag{19}$$

The solution of the homogeneous equation is a damped oscillation which disappears after some time. We are left with the particular solution of the form $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Inserting this into the differential equation and separating cosine and sine terms gives

$$(\omega_r^2 - \omega^2)A + \frac{\omega_r\omega}{Q}B = 0 \text{ and } (\omega_r^2 - \omega^2)B - \frac{\omega_r\omega}{Q}A = -\frac{\omega_r\omega R_s}{Q}\hat{I}. \tag{20}$$

The voltage induced by the harmonic excitation of the resonator becomes

$$V(t) = \hat{I}R_s\frac{\cos(\omega t) - Q\frac{\omega_r-\omega}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}. \tag{21}$$

The voltage has a cosine term which is *in phase* with the exciting current. It can absorb energy and is called *resistive* term. The sine term of the voltage is *out of phase* with the exciting current and does not absorb energy, it is called *reactive*. The ratio between the voltage and current is the *impedance*. It is a *function of frequency* ω and has a resistive part $Z_r(\omega)$ and a reactive part $Z_i(\omega)$

$$Z_r(\omega) = R_s\frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}\ ,\ Z_i(\omega) = (\pm)R_s\frac{Q\frac{\omega_r^2-\omega^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}. \tag{22}$$

2.4 Complex notation

We have used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{e^{j\omega t} + e^{-j\omega t}}{2} \text{ with } 0 \le \omega \le \infty. \tag{23}$$

It is often more convenient to use a complex notation

$$It) = \hat{I}e^{j\omega t} \text{ with } -\infty \le \omega \le \infty \tag{24}$$

which leads to more compact expressions. Taking the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 = \frac{\omega_r R_s}{Q}\dot{I} \tag{25}$$

of the excited resonance with $I(t) = \hat{I}\exp(j\omega t)$ and seeking a solution of the form $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex, one gets

$$-\omega^2 V_0 e^{j\omega t} + j\frac{\omega_r - \omega}{Q} V_0 e^{j\omega t} + \omega_r^2 V_0 e^{j\omega t} = j\frac{\omega_r \omega R_s}{Q}\hat{I}e^{j\omega t} \tag{26}$$

and for the impedance which is defined as the ratio V/I,

$$Z(\omega) = \frac{V_0}{\hat{I}} = R_s \frac{j\frac{\omega_r\omega}{Q}}{\omega_r^2 - \omega^2 + jQ\frac{\omega_r\omega}{Q}} = R_s \frac{1 - j\frac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2 - \omega_r^2}{\omega\omega_r}\right)^2}. \tag{27}$$

For a large quality factor the impedance is only large for $\omega \approx \omega_r$ or $|\omega - \omega_r|/\omega_r = |\Delta\omega|\omega_r \ll 1$ and can be simplified

$$Z(\omega) \approx R_s \frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}. \tag{28}$$

The resonator impedance has some specific properties:

$$\begin{aligned} \omega &= \omega_r \rightarrow Z_r(\omega_r) \text{ has a maximum, } Z_i(\omega_r) = 0 \\ |\omega| &< \omega_r \rightarrow Z_i(\omega) > \omega_r \text{ (inductive)} \\ |\omega| &> 0 \rightarrow Z_i(\omega) < 0 \text{ (capacitive)} \end{aligned} \tag{29}$$

and some properties which apply to any impedance or wake potential

$$Z_r(\omega) = Z_r(-\omega)\ ,\ Z_i(\omega) = -Z_i(-\omega),$$

$$Z(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt,\ \ Z(\omega) = \text{Fourier transform of } G(t), \text{Fig.5}, \tag{30}$$

$$t < 0 \rightarrow G(t) = 0 \text{ no fields before particle arrives.} \tag{31}$$

Caution: sometimes one uses $I(t) = \hat{I}e^{-i\omega t}$ instead of $I(t) = \hat{I}e^{j\omega t}$, this reverses the sign $Z_i(\omega)$.

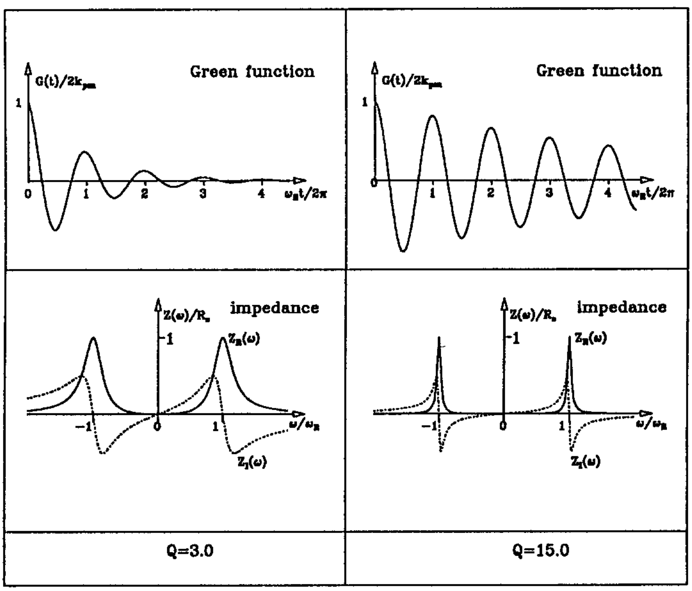


Figure 5: Green function and impedance of a resonance

3 Interaction of a stationary bunch with an impedance

3.1 Spectrum of a stationary bunch

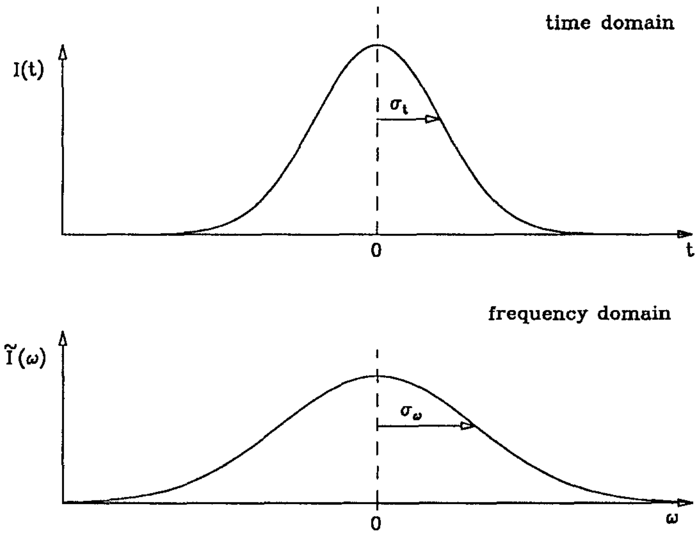


Figure 6: Single passage of a bunch in time and frequency domain

We consider a single traversal at the time $t = 0$ of a bunch having the form $I(t)$ which we assume for convenience to be symmetric in t. The spectrum of this single bunch traversal is given by the *Fourier transform*, Fig. 6,

$$\tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) \mathrm{e}^{-j\omega t} dt. \tag{32}$$

Had this bunch passed by the observer at an earlier time T its current and spectrum would be of the form $I_T(t) = I(t-T)$ and

$$\begin{aligned} \tilde{I}_T(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t-T) \mathrm{e}^{-j\omega t} dt \\ &= \frac{\mathrm{e}^{-j\omega T}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t-T) \mathrm{e}^{-j\omega(t-T)} d(t-T) = \mathrm{e}^{-j\omega T} \tilde{I}(\omega) \, . \end{aligned}$$

A shift of the time domain function produces a phase factor in the Fourier transform (*Fourier shift theorem*).

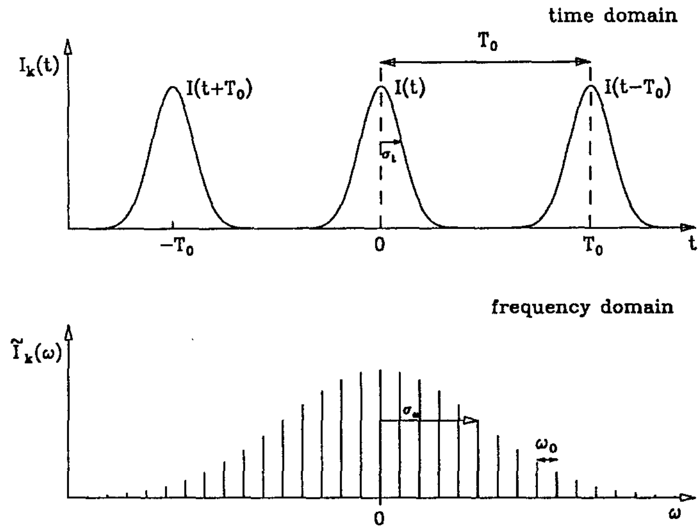


Figure 7: Multiple passage of a bunch in time and frequency domain

Next we take the case of a bunch circulating in a storage ring with revolution time T_0. It represents a current

$$I_k(t) = \sum_{k=-\infty}^{k=\infty} I(t - kT_0). \tag{33}$$

The spectrum is obtained using the shift theorem

$$\tilde{I}_k(\omega) = \sum_{k=-\infty}^{\infty} e^{-j\omega kT_0}\, \tilde{I}(\omega)\,. \tag{34}$$

Using the relation

$$\sum_{k=-\infty}^{\infty} e^{-jkx} = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p) \text{ and } \delta(ax) = \frac{1}{a}\delta x \tag{35}$$

we get

$$\sum_{k=-\infty}^{\infty} e^{-jk\omega T_0} = 2\pi \sum_{p=-\infty}^{\infty} \delta(\omega T_0 - 2\pi p) = \frac{2\pi}{T_0} \sum_{p=-\infty}^{\infty} \delta(\omega - p\omega_0), \tag{36}$$

where $\omega_0 = 2\pi/T_0$ is the revolution frequency. The Fourier transformed current is

$$\tilde{I}_k(\omega) = \omega_0 \sum_{p=-\infty}^{\infty} \tilde{I}(\omega)\delta(\omega - p\omega_0). \tag{37}$$

An inverse Fourier transform gives the current in time domain, Fig. 7,

$$I_k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{I}_k(\omega) e^{j\omega t} d\omega = \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\omega - p\omega_0)\tilde{I}(\omega) e^{j\omega t} d\omega \tag{38}$$

$$I_k(t) = \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \tilde{I}(p\omega_0) e^{jp\omega_0 t}. \tag{39}$$

Since we assumed $I(t)$ to be symmetric we have $\tilde{I}(p\omega_0)$ is real and $\tilde{I}(-p\omega_0) = \tilde{I}(p\omega_0)$ so that combining positive and negative frequency terms gives

$$I_k(t) = I_0 + \frac{2\omega_0}{\sqrt{2\pi}} \sum_{p=1}^{\infty} \tilde{I}(p\omega_0)\cos(p\omega_0 t). \tag{40}$$

We used here a Fourier transform in view of later applications. Since the bunch current $I_k(t)$ is a periodic function it would have been easier to develop it into a Fourier series

$$I_k(t) = I_0 + \sum_{1}^{\infty} I_p \cos(p\omega_0 t) \ \text{ with } \ I_p = \frac{2\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0). \tag{41}$$

For low frequencies $I_p \approx 2I_0$.

3.2 Voltage induced by the stationary bunch

In the presence of a cavity resonance or any *general impedance* $Z(\omega)$ the circulating stationary bunch induces a voltage in frequency domain

$$\tilde{V}_k(\omega) = \tilde{I}_k(\omega)Z(\omega) = \omega_0 \sum_{p=-\infty}^{\infty} \tilde{I}(\omega)\delta(\omega - p\omega_0)Z(\omega) \tag{42}$$

and in time domain

$$\begin{aligned} V_k(t) &= \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{I}(\omega)\delta(\omega - p\omega_0)Z(\omega)e^{j\omega t}d\omega \\ &= \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \tilde{I}(p\omega_0)Z(p\omega_0)e^{jp\omega_0 t} \end{aligned}$$

or, by combining positive and negative frequencies and observing the symmetry conditions $Z_r(-\omega) = Z_r(\omega)$, $Z_i(-\omega) = -Z_i(\omega)$ and the fact that $Z(0) = 0$

$$V_k(t) = \frac{2\omega_0}{\sqrt{2\pi}} \sum_{p=1}^{\infty} \tilde{I}(p\omega_0)\left(Z_r(p\omega_0)\cos(p\omega_0 t) - Z_i(p\omega_0)\sin(p\omega_0 t)\right). \tag{43}$$

3.3 Energy loss per turn of a stationary circulating bunch

The energy lost by the circulating stationary bunch in one turn due to the impedance $Z(\omega)$ is

$$W_b = \int_0^{T_0} I_k(t)V_k(t)dt. \tag{44}$$

From Sections 3.1 and 3.2

$$I_k(t) = \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \tilde{I}(p\omega_0)e^{jp\omega_0 t} \tag{45}$$

$$V_k(t) = \frac{\omega_0}{\sqrt{2\pi}} \sum_{p'=-\infty}^{\infty} \tilde{I}(p'\omega_0)Z(p'\omega_0)e^{jp'\omega_0 t} \tag{46}$$

so that

$$W_b = \frac{\omega_0^2}{2\pi} \sum_{p=-\infty}^{\infty} \sum_{p'=-\infty}^{\infty} \tilde{I}(p\omega_0)\tilde{I}(p'\omega_0)Z(p'\omega_0) \int_0^{T_0} e^{j\omega_0(p+p')t}dt. \tag{47}$$

The integral

$$\int_0^{T_0} e^{j\omega_0(p+p')t}dt = \int_0^{T_0} \left(\cos(\omega_0(p+p')t) + j\sin(\omega_0(p+p')t)\right)dt \tag{48}$$

vanishes except for $p+p' = 0$ in which case it takes the value $T_0 = 2p/\omega_0$. We have therefore $p = -p'$

$$W_b = \omega_0 \sum_{p=-\infty}^{\infty} \left|\tilde{I}(p\omega_0)\right|^2 Z(p\omega_0) = 2\omega_0 \sum_{1}^{\infty} \left|\tilde{I}(p\omega_0)\right|^2 Z_r(p\omega_0) \tag{49}$$

where we used $\tilde{I}(\omega)\tilde{I}(-\omega) = \tilde{I}(\omega)\tilde{I}^*(\omega)) = |\tilde{I}(\omega)|^2$, which is true for the Fourier transform of a real function $I(t)$, and the symmetry relations $Z_r(\omega) = Z_r(-\omega)$, $Z_i(\omega) = -Z_i(-\omega)$.

This is the energy loss of the whole bunch. The average loss per particle U is

$$U = \frac{W_b}{N} = \frac{\omega_0 e U_b}{2\pi I_0} \tag{50}$$

where N is the number of particles and I_0 the current of the bunch.

4 Interaction of an oscillating bunch with a cavity

4.1 Review of the longitudinal dynamics

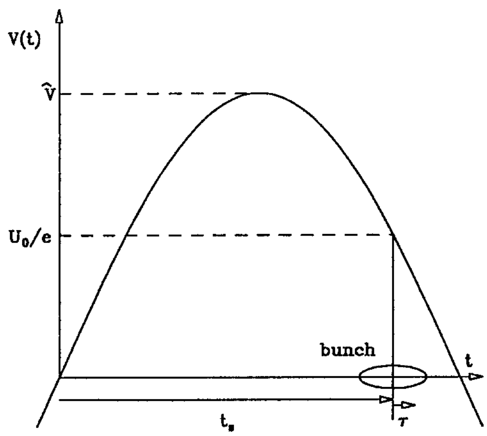


Figure 8: Longitudinal beam dynamics

A particle with a momentum deviation Δp has a different closed orbit which is displaced by $\Delta x = \alpha \Delta p/p$. As a result the orbit length L, the revolution time T_0 and the revolution frequency ω_0 are changed:

$$\frac{\Delta L}{L} = \alpha\frac{\Delta p}{p}\ , \ \ \frac{\Delta\omega_0}{\omega_0} = -\frac{\Delta T_0}{T_0} = -\left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = -\eta\frac{\Delta p}{p} \tag{51}$$

with $\eta = \alpha 1/\gamma^2$. There is a transition energy $E_T = m_0 c^2 \gamma_T$ with $\gamma_T = 1/\alpha^2$ for which the dependence of the revolution frequency on momentum (or energy) changes sign:

$$E > E_T \ \rightarrow\ \frac{1}{\gamma^2} < \alpha \ \rightarrow\ \eta > 1 \ \rightarrow\ \omega_0 \,\text{decreases with}\, \Delta E \tag{52}$$

$$E < E_T \ \rightarrow\ \frac{1}{\gamma^2} > \alpha \ \rightarrow\ \eta < 1 \ \rightarrow\ \omega_0 \,\text{increases with}\, \Delta E\,. \tag{53}$$

Many machines work far above transition energy and $\eta \approx \alpha$. We will assume that the particles are ultra relativistic in which case $\Delta p/p \approx \Delta E/E = \epsilon$. In the presence of an RF system, of synchrotron radiation loss U_s and of the loss due to impedance U, a circulating particle has an energy gain δE per turn

$$\delta E = e\hat{V}\sin(h\omega_0(t_s+\tau) - U_s(E) - U(t) \tag{54}$$

with t_s being the synchronous arrival time of the particle in the cavity and τ the deviation from it, and U_s the energy loss per turn due to synchrotron radiation. Introducing the synchronous phase angle $\phi_s = h\omega_0 t_s$ and assuming the $\tau \ll T_0$, Fig. 8, we get

$$\delta E \approx e\hat{V}\left[\sin(\phi_s) + \cos(\phi_s)h\omega_0\tau\right] - U_0 - \frac{dU_s}{dE}\Delta E - \frac{dU}{dt}\tau. \tag{55}$$

The energy gain per turn is very small $\delta E \ll E$ and we can make a smooth approximation

$$\frac{\Delta E}{E} = \dot{\epsilon}T_0 = \dot{\epsilon}\frac{2\pi}{\omega_0} \tag{56}$$

$$\dot{\epsilon} = \frac{e\hat{V}\sin\phi_s\omega_0}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U_0}{E} - \frac{\omega_0}{2\pi}\frac{dU_s}{dE}\epsilon - \frac{1}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\tau. \tag{57}$$

To have equilibrium for the synchronous particle $\epsilon = 0$, $\tau = 0$ we have $U_0 = e\hat{V}\sin\phi_s$. Using $\dot{\tau} = \frac{\omega_0}{2\pi}\Delta T_0 = \eta\epsilon$ we get

$$\begin{aligned}
\dot{\epsilon} &= \omega_0^2\frac{he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{dU_s}{dE}\epsilon - \frac{1}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\tau \\
\dot{\tau} &= \eta\epsilon.
\end{aligned}$$

These two first-order differential equations can be combined into a second-order one

$$\ddot{\epsilon} + \frac{dU_s}{dE}\frac{\omega_0}{2\pi}\dot{\epsilon} - \frac{\omega_0^2 h\eta e\hat{V}\cos\phi_s}{2\pi E}\epsilon - \frac{\eta}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\epsilon = 0 \tag{58}$$

which is the equation of a damped oscillation. Using

$$\omega_{s0}^2 = -\omega_0^2\frac{h\eta e\hat{V}\cos\phi_s}{2\pi E}\ ,\ \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU_s}{dE}, \tag{59}$$

seeking a solution of the form $\exp(j\omega t)$ and assuming $\alpha_s \ll \omega_{s0}$ we get

$$-\omega^2 + j\omega\alpha_s + (\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta}{E}\frac{dU}{dt}) = 0 \tag{60}$$

$$\omega = j\alpha_s \pm \sqrt{(\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta}{E}\frac{dU}{dt}) - \alpha_s^2} \approx j\alpha_s \pm (\omega_{s0} + \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta}{\omega_{s0}E}\frac{dU}{dt}). \tag{61}$$

Calling

$$\Delta\omega_i = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta}{\omega_{s0}E}\frac{dU}{dt} \tag{62}$$

gives

$$\epsilon = A\left(e^{(-\alpha_s + j(\omega_{s0}+\Delta\omega_i)t} + Be^{(-\alpha_s - j(\omega_{s0}+\Delta_i)t}\right). \tag{63}$$

For the initial conditions $\epsilon(t) = \hat{\epsilon}$, $\dot{\epsilon}(0) = -\alpha_s\hat{\epsilon}$ we get $A = B = \hat{\epsilon}/2$ and

$$\epsilon(t) = \hat{\epsilon}e^{-\alpha_s t}\cos((\omega_{s0} + \Delta\omega_i)t). \tag{64}$$

In order to get a stable oscillation we need

$$E > E_T \rightarrow \cos\phi_s < 0\ ,\ E < E_T \rightarrow \cos\phi_s > 0. \tag{65}$$

4.2 Qualitative treatment of the Robinson instability

The most important effect of the interaction between a longitudinally oscillating bunch and a cavity is the so-called Robinson instability [1] which is treated here in some detail since it can be generalized to describe all multi-turn instabilities in storage rings. We start with a qualitative treatment of the Robinson instability by considering a single bunch circulating in a storage ring and exciting a cavity resonance with resonance frequency ω_r and impedance $Z(\omega)$ of which we consider only the resistive part Z_r.

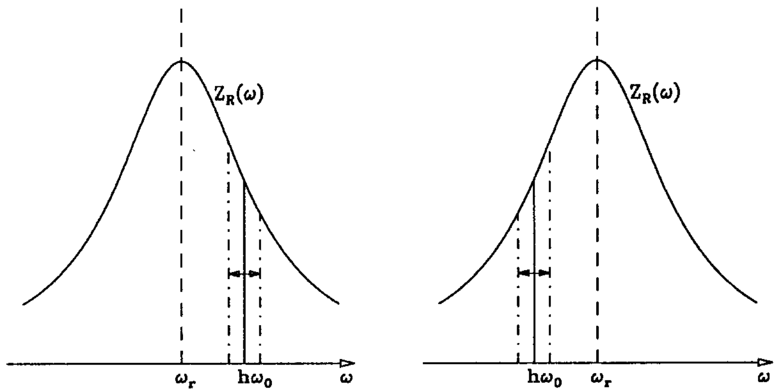


Figure 9: Qualitative treatment of the Robinson instability

The revolution frequency ω_s of the circulating bunch depends on its energy deviation ΔE

$$\frac{\Delta\omega_0}{\omega_0} = -\alpha\frac{\Delta E}{E} \text{ or } \omega_0 = p\omega_{0c}\left(1 - \alpha\frac{\Delta E}{E}\right). \tag{66}$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its revolution frequency is modulated. *Above transition* the revolution frequency ω_0 is *small* when the *energy is high* and ω_0 is *large* when the *energy is small.* If the cavity is tuned to a resonant frequency being slightly smaller than the RF frequency $\omega_r < p\omega_0$, Fig. 9 left, the bunch sees a higher impedance and *loses more energy* when it has an *energy excess* and it *loses less energy* when it has a *lack of energy.* This leads to a *damping* of the oscillation. If $\omega_r > p\omega_0$ this is reversed, Fig. 9 right, and leads to an *instability.* Below transition energy the dependence of the revolution frequency is reversed, which changes the stability criterion.

4.3 Semi-quantitative treatment of the Robinson instability

We consider a narrow band cavity with a circulating bunch as before. The bunch executes a synchrotron oscillation which is approximately described as $\tau = \hat{\tau}\cos(\omega_s t)$. This will produce sidebands to the revolution frequency harmonics of the bunch. Taking now only the revolution harmonic h, i.e. the RF frequency, we have

$$\begin{aligned} I_k(t) &= I_h \cos(h\omega_0 t - h\omega_0\hat{\tau}\cos(\omega_s t)) \\ &\approx I_h\left(\cos(h\omega_0 t) + \sin(h\omega_0 t)h\omega_0\sin(h\omega_s t)\right) \end{aligned}$$

for $\hat{\tau} \ll T_0$, or

$$I_k(t) = I_h\left[\cos(h\omega_0 t) + \frac{h\omega_0\hat{\tau}}{2}\left(\sin((h\omega_0 + \omega_s)t) + \sin((h\omega_0 - \omega_s)t)\right)\right]. \tag{67}$$

We calculate the voltage induced by the bunch in the cavity and take *only the resistive part* of the impedance. Using

$$Z^+ = Z(h\omega_0 + \omega_s)\ ,\ Z^- = Z(h\omega_0 - \omega_s)\ ,\ Z^0 = Z(h\omega_0) \tag{68}$$

we get

$$\begin{aligned} V_k(t) &= I_h\,[\,Z_r^0\cos(h\omega_0 t) \\ &+ \frac{h\omega_0}{2}\left(Z_r^+\sin((h\omega_0+\omega_s)t) + Z_r^-\sin((h\omega_0-\omega_s)t)\right)] \\ &= I_h\,[\,Z_r^0\cos(h\omega_0 t) \\ &+ \frac{h\omega_0\hat{\tau}}{2}(\,Z_r^+(\sin(h\omega_0 t)\cos(\omega_s t) + \cos(h\omega_0 t)\sin(\omega_s t)) \\ &+ Z_r^-(\sin(h\omega_0 t)\cos(\omega_s t) - \cos(h\omega_0 t)\sin(\omega_s t))\,)\,]\,. \end{aligned}$$

We use

$$\tau = \hat{\tau}\cos(\omega_s t)\ ,\ \dot{\tau} = -\omega_s\hat{\tau}\sin(\omega_s t). \tag{69}$$

so that

$$\begin{aligned} V_k(t) &= I_h\,[\,Z_r^0\cos(h\omega_0 t) \\ &+ \frac{h\omega_0}{2}\left(Z_r^+(\sin(h\omega_0 t)\tau - \cos(h\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right) \\ &+ \frac{h\omega_0}{2}\left(Z_r^-(\sin(h\omega_0 t)\tau + \cos(h\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right)\,]\,. \end{aligned}$$

We calculate now the energy loss per turn

$$W_b = \int_0^{T_0} V_k(t)I_k(t)dt = \frac{I_h^2 T_0}{2}\left[Z_r^0 - \frac{h\omega_0}{2}\frac{\dot{\tau}}{\omega_s}(Z^+ - Z^-)\right]. \tag{70}$$

The average loss per particle is

$$U = \frac{W_b}{N} = W_b\frac{e\omega_0}{2\pi I_0} = \frac{eI_h^2 Z_r^0}{2} - \frac{eI_h^2 h\omega_0}{4I_0\omega_s}(Z_r^+ - Z_r^-)\dot{\tau} \tag{71}$$

and with the relation $\dot{\tau} = \eta\epsilon$ we get

$$U_0 = \frac{eI_h^2 Z_r^0}{2}\ \text{and}\ \frac{dU}{dE} = \frac{eh\eta\omega_0 I_h^2(Z_r^+ - Z_r^-)}{4I_0 E\omega_s}. \tag{72}$$

With

$$\epsilon = \hat{\epsilon}e^{-\alpha_s t}\cos(\omega_s t + \phi)\ ,\alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU}{d\epsilon} \tag{73}$$

we get for the damping or growth rate of the Robinson instability

$$\alpha_s = \frac{\omega_0^2 h\eta e I_h^2(Z_r^+ - Z_r^-)}{16\pi\omega_s I_0 E} = \frac{\omega_s I_h^2(Z_r^+ - Z_r^-)}{8I_0\hat{V}\cos(\phi_s)} \tag{74}$$

where we used

$$\omega_{s0}^2 = -\omega_0^2\frac{h\eta e\hat{V}\cos\phi_s}{2\pi E}. \tag{75}$$

Since the bunch length is usually much shorter than the RF wavelength we have $I_h \approx 2I_0$

$$\alpha_s \approx \frac{\omega_s I_0(Z_r^+ - Z_r^-)}{2\hat{V}\cos(\phi_s)}. \tag{76}$$

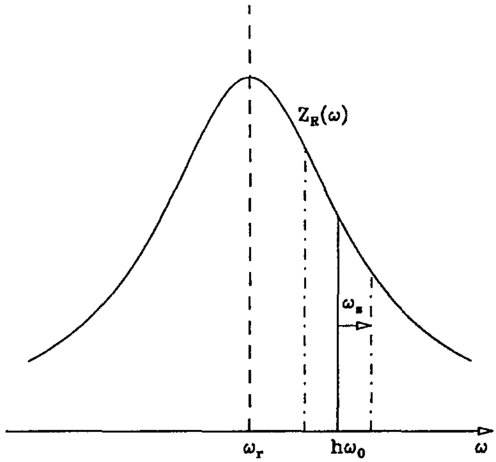


Figure 10: Semi-quantitative treatment of the Robinson instability

The growth rate of the Robinson instability is given by the difference of the resistive impedance at the upper and lower synchrotron side band. Above transition energy we have $\cos\phi_s < 0$ and $\alpha_s > 0$, i.e. stability if $Z_r^- > Z_r^+$ as we found already from qualitative arguments, Fig. 10. We neglected here the effect of the reactive part of the impedance. It leads to a shift of the synchrotron frequency [6].

4.4 Discussion and generalization of the Robinson instability

We have derived the Robinson instability for the case of a single bunch and a single, narrow band and relatively weak resonance. We will here briefely discuss the way this instability can be extended to cover more general cases.

A more general impedance will cover not just a single revolution harmonic with the two synchrotron oscillation sidebands but many such frequency lines. In this case the voltage induced in the impedance by each such line contained in the spactrum of the bunch current has to be considered. The growth rate will no longer be given by the difference between the impedance at the upper and lower synchrotron side band but between the sums of the impedance times the spectral power taken at each upper and each lower synchrotron side band contained in the spectrum of the oscillating bunch.

This can be extended to the case of many bunches [2, 3]. With M equidistant bunches in the machine we have M different modes of oscillation each having a different phase between the oscillations executed by adjacient bunches. The spectrum of each such coupled bunch mode has synchrotron side bands at distinct revolution harmonics. In calculating the stability of a certain coupled bunch mode we have to sum over these side bands.

So far we considered only dipole oscillations where the bunch makes a rigid oscillation around the nominal phase without changing the form. There are higher modes of oscillation, called bunch shape oscillations, which can be classified as quadrupole ($m = 2$), sextupole ($m = 3$), octupole ($m = 4$), etc. modes. Each mode has a spectrum with side bands at $m\omega_s$ from the revolution harmonics. Again, to calculate the stability of these modes we have to sum over these side bands.

We have so far assumed that the effect of the impedance is relatively weak such that the changes of the synchrotron frequency and the growth rate of the instability are small compared to the synchrotron frequency itself. For very narrow band cavities with high shunt impedance, e.g. superconducting cavities, this might no longer be true. In this case we have to evaluate the impedance not at the unperturbed side band ω_{s0} but at the shiftet synchrotron frequency ω_s. Furthermore, if we are interested in the growth rate we have to consider the cavity impedance for a growing oscillation which is different as soon as the growth time of the oscillation becomes comparable to the filling time of the cavity. Taking all this into account one arrives at a 4th order equation for the shifted synchrotron frequency and the growth rate for which a more general stability criterion can be derived, often called the second Robinson instability [1].

We have so far considered stability for the case of infinitesimally small oscillations and calculated their growth or damping time. If the oscillation amplitude becomes large some non-linear effects should be considered. The modulation index of the phase oscillation will become large leading to side bands at twice the synchrotron frequency. They have to be included when forming the sum over the impedance contribution. This can lead to a situation where the beam is unstable for small oscillation amplitudes but becomes stable again at large amplitudes. Such cases represent themselves in practice as bunches oscillating with finite but more or less constant amplitudes [4, 5].

5 Transverse Stability of a Coasting Beam and Landau Damping

5.1 Introduction

We investigate now the transverse stability of a coasting, i.e. an unbunched, beam which has no longitudinal time structure. Such beams can only exist over longer time for particles and conditions where the energy loss due to synchrotron radiation is negligible. The application of the following treatment is therefore somewhat limited but, on the other hand, it is relatively simple and tranparent. We use the case of the transverse stability of such a coasting beam mainly to explain the concept of Landau damping which is much more complicated for bunched beams. We follow here basically the method used in earlier Accelerator Schools [6, 7].

5.2 Transverse oscillation modes of a coasting beam

We consider now a coasting (unbunched) beam circulating without energy loss or gain in a storage ring. We assume that this beam has a central momentum p_0 with a corresponding revolution frequency ω_{0c}. The particles themselves have a distribution in momentum with deviations Δp and

$$\Delta\omega_0 = -\omega_{0c}\eta\frac{\Delta p}{p} \tag{77}$$

from the central values with distribution function

$$F_p(\Delta p) = \frac{1}{N}\frac{dN}{dp}, \tag{78}$$

where $\eta = \alpha - 1/\gamma^2$, $\alpha =$ momentum compaction and N is the number of particles in the beam.

The beam executes at the same time a vertical betatron oscillation such that the motion of each particle consists of a rotation $\theta = \omega_0 t$, and a vertical oscillation $y = \hat{y}\cos(Q\omega_0 t)$, see Fig. 6. This beam as a whole can execute such a motion in a very large number of modes as long as we do not fix a phase relation between the betatron oscillation of the individual particles. We are, of course, interested in modes which are very simple, e.g. where all the particles go up and down together or with a simple phase relation. Such simple modes contain only relatively low frequencies and are more likely to interact with realistic impedances of the beam surroundings. We classify the modes as closed waves with a different number n of undulations around the circumference. They can be described as

$$y_n = \hat{y}\cos(n\theta - \omega t)\,,\ or\ \ y_n = \hat{y}e^{i(n\theta-\omega t)}. \tag{79}$$

For $n = 0$ all particles move together. If the motion is frozen by fixing the time t, the mode number n gives the number of waves around the ring ($n = 4$ for Fig. 11).

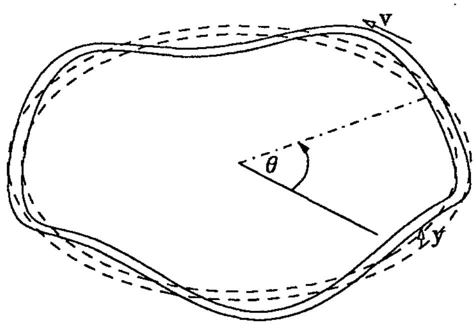


Figure 11: Oscillation modes of a coasting beam

The mode number n can take positive or negative values depending on a phase advance or a phase lag of the oscillation executed by a particle in front of a reference particle. These two classes of modes are often called fast or forward, and slow or backward, waves. We now wish to determine the frequency ω which is seen by a stationary observer at a fixed location θ. We know that the vertical oscillation frequency is $Q\omega_0$ for an individual particle. Such a particle executes the azimuthal motion $\theta = \theta_0 + \omega_0 t$. This gives

$$e^{i(n\theta_0 + n\omega_0 t - \omega t)} = e^{n\theta_0 - iQ\omega_0 t} \tag{80}$$

or

$$\omega = (n + Q)\omega_0 = \omega_\beta, \tag{81}$$

where n goes through positive and negative integers. It is convenient to divide these modes into fast waves

$$\omega_{\beta f} = (n_f + Q)\omega_0\,, \;\; n_f > -Q \tag{82}$$

and slow waves

$$\omega_{\beta s} = (n_s - Q)\omega_0\,, \;\; n_s > Q. \tag{83}$$

The spread in momentum of the particles in the beam results in a spread in the betatron frequency ω_β through two effects. The revolution frequency ω_0 depends on Δp

$$\Delta\omega_0 = -\omega_0 \eta \frac{\Delta p}{p}\,, \;\; \eta = \alpha - \frac{1}{\gamma^2}. \tag{84}$$

Furthermore the betatron tune Q depends on the momentum through the chromaticity ξ

$$\Delta Q = \xi Q \frac{\Delta p}{p} = Q' \frac{\Delta p}{p}\,, \;\; \xi = \frac{\Delta Q/Q}{\Delta p/p}. \tag{85}$$

This leads to a spread in betatron frequency observed at a stationary location

$$\Delta\omega_{\beta f} = (Q' - \eta(n_f + Q))\omega_0 \frac{\Delta p}{p}\,, \;\; \Delta\omega_{\beta s} = (Q' - \eta(n_s - Q))\omega_0 \frac{\Delta p}{p}. \tag{86}$$

5.3 Response of the beam to a transverse excitation

We consider now an experiment where a periodic transverse force is applied to the beam by an electromagnetic device of length L_G. This force results in an acceleration G averaged over one revolution

$$G = \frac{F}{m} = \hat{G}e^{-i\omega t} = \frac{e[\vec{E} + [\vec{\beta}c \times \vec{B}]]_T}{m_0\gamma}\frac{L_G}{2\pi R}. \tag{87}$$

The equation of motion for a particle with betatron wave number Q and revolution frequency ω is

$$\frac{d^2y}{dt^2} + \omega_0^2 Q^2 y = \hat{G}e^{-i\omega t}. \tag{88}$$

We are seeking a solution where a string of particles appear to a stationary observer as

$$y = \hat{y}e^{i(n\theta - \omega t)}. \tag{89}$$

Using this and the relation between absolute and partial differentiation we get

$$\frac{dy}{dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial \theta}\dot{\theta} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial \theta}\omega_0 = i(n\omega_0 - \omega)y \tag{90}$$

and the equation of motion becomes

$$[-(n\omega_0 - \omega)^2 + \omega_o^2 Q|s^2]\hat{y}e^{i(n\theta - \omega t)} = \hat{G}e^{-i\omega t}. \tag{91}$$

We solve this for $\hat{y}$ and assume that $\theta = 0$ which means that the oscillation is observed close to the exciter

$$\hat{y} = \frac{\hat{G}}{\omega_0^2 Q^2 - (n\omega_0 - \omega)^2} = \frac{-G_0}{(\omega - \omega_0(n+Q))(\omega - \omega_0(n-Q))} \tag{92}$$

$$\frac{\hat{y}}{\hat{G}} = \frac{-1}{(\omega_{\beta f} - \omega)(\omega_{\beta s} - \omega)}. \tag{93}$$

This is the response of a single particle. It is only large if the exciting frequency ω is either close to the frequency $\omega_{\beta s}$ of the fast, or to the frequency $\omega_{\beta s}$ of the slow, wave. We will approximate for these two cases and get for the single particle responses

$$\left(\frac{\hat{y}}{\hat{G}}\right)_f \sim \frac{-1}{2Q\omega_0}\frac{1}{(\omega_{\beta f} - \omega)}, \quad \left(\frac{\hat{y}}{\hat{G}}\right)_s \sim \frac{1}{2Q\omega_0}\frac{1}{(\omega_{\beta s} - \omega)}. \tag{94}$$

This approximation takes only the response to the mode n (being closest in frequency) while the contributions due to the other modes are ignored [8]. In this form the single particle response is the same as that for the case of a set of oscillators used earlier. Therefore, we can easily find the response for the centre of charge for the fast wave case by an integration over the single particle response weighted with the distribution $f(\omega_{\beta f})$ which is normalized such that $\int f(\omega_{\beta f})d\omega_{\beta f} - 2\pi$

$$< \hat{y} >_f = \int f(\omega_{\beta f})\hat{y}(\omega_{\beta f})d\omega_{\beta f} = -\frac{\hat{G}}{2Q\omega_0}\int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega}d\omega_{\beta f} = \frac{\hat{G}}{2Q\omega_0}\left[PV\int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega}d\omega_{\beta f} \pm i\pi f(\omega)\right]. \tag{95}$$

This integral goes over a pole at $\omega = \omega_{\beta f}$ where the denominator vanishes. Such an integral has two parts, a principle value "PV" which is obtained by excluding the pole with a small gap and an imaginary residue the sign of which is ambiguous since we did not speczify the initial conditions. The physical meaning of the two parts can be made more clear by calculating the beam response in velocity $< \dot{y} >$ of the centre-of-mass motion

$$< \dot{y} >_s = -i\omega < y > = -\frac{\hat{G}\omega e^{-i\omega t}}{2Q\omega_0}[\pi f(\omega) - iPV\int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega}d\omega_{\beta f}]. \tag{96}$$

The corresponding expression for the slow wave is obtained by changing the sign and replacing $\omega_{\beta f}$ by $\omega_{\beta s}$. In both cases the limits of the integration are such that only the side band in question is covered. This is the response of a coasting beam to a transverse harmonic excitation with a frequency ω being close to the one of a betatron side band ω_β. In our approximation the integration has to cover just such a single side band and we neglect the influence of other side bands. This response has a real part which is — apart from a common factor — equal to $\pi f(\omega)$. The fact that this part is real means that for this term the velocity is *in phase* with the exciting acceleration and can absorb energy. It can therefore lead to damping of the centre-of-mass oscillation. The second term with the principle value integral is imaginary which means that the velocity is out of phase compared to the acceleration and no energy is exchanged. For this reason the two terms are also called resistive and reactive response. The resistive term can lead to a damping called Landau damping. This effect rests essentially on the existance of the residue. Since this is not very transparent we will derive the beam response again using real notation only but specifying the initial conditions clearly.

5.4 Time evolution of the response using real notation

We saw that the resistive part of the beam response to a harmonic excitation is due to the residue of the integral. Since it is this term which leads to Landau damping we would like to better understand the underlying physics. For this purpose we now discuss the time evolution of the response using real notation [6]. This leads to derivations which are more lengthy but more transparent. The differential equation describing the excitation of the beam is, in analogy with Eq. (96),

$$\frac{d^2y}{dt^2} + \omega_\beta^2 y = \hat{G}\sin(\omega t). \tag{97}$$

This equation has a homogeneous solution

$$y_h = A\sin(\omega_\beta t) + B\cos(\omega_\beta t) \tag{98}$$

and a particular solution

$$y_p = \frac{\hat{G}}{\omega_\beta^2 - \omega^2}\sin(\omega t). \tag{99}$$

The general solution is a combination of the two,

$$y = A\sin(\omega_\beta t) + B\cos(\omega_\beta t) + \frac{\hat{G}}{\omega_\beta^2 - \omega 2}\sin(\omega t). \tag{100}$$

To determine the integration constants A and B we have to specify the initial conditions. We take the case where the oscillators are at rest until the time $t = 0$ when the excitation starts,

$$y(0) = \dot{y}(0) = 0 \;\rightarrow\; A = -\frac{\omega}{\omega_\beta}\frac{\hat{G}}{(\omega_\beta^2 - \omega 2)}\;;\;\; B = 0. \tag{101}$$

This gives for the general solution

$$y = \frac{\hat{G}}{\omega_\beta 2 - \omega 2}\left(\sin(\omega t) - \frac{\omega}{\omega_\beta}\sin(\omega_\beta t)\right) = \frac{\hat{G}}{2\omega}\left(\frac{1}{(\omega_\beta - \omega)} - \frac{1}{\omega_\beta + \omega}\right)\left(\sin(\omega t) - \frac{\omega}{\omega_\beta}\sin(\omega_\beta t)\right). \tag{102}$$

We differentiate with respect to t to get the velocity of the single oscillator response

$$\dot{y} = \frac{\hat{G}}{2}\left(\frac{\cos(\omega t) - \cos(\omega_\beta t)}{(\omega_\beta - \omega)} - \frac{\cos(\omega t) - \cos(\omega_\beta t)}{\omega_\beta + \omega}\right) \tag{103}$$

and rewrite this equation by substituting ω_β with $\omega_\beta = \omega + (\omega_\beta - \omega)$ in the first term and $\omega_\beta = (\omega_\beta + \omega) - \omega$ in the second term of the parenthesis,

$$\begin{aligned} \dot{y} &= \frac{\hat{G}}{2}\left[\cos(\omega t)\left(\frac{1-\cos((\omega_\beta-\omega)t)}{(\omega_\beta-\omega)} - \frac{1-\cos((\omega_\beta+\omega)t)}{\omega_\beta+\omega}\right)\right. \\ &+ \left.\sin(\omega t)\left(\frac{\sin((\omega_\beta-\omega)t)}{(\omega_\beta-\omega)} + \frac{\sin((\omega_\beta+\omega)t)}{\omega_\beta+\omega}\right)\right]. \end{aligned} \tag{104}$$

The above equation gives the velocity response for a single oscillator with resonant frequency ω_β. For the first term in the square bracket velocity and acceleration are basically out of phase while for the second term they are in phase. This statement is not exact since both terms also contain an oscillatory term with frequency $(\omega_\beta - \omega)$ which has to be discussed further. Before we integrate to obtain the centre-of-mass response we discuss these two components of the single oscillator response Eq. (104). We concentrate on a region in the vicinity of the exciting positive frequency ω, i.e. $\omega_\beta \approx \omega$ where the first term inside each of the two round brackets is dominant. A corresponding discussion for $\omega_\beta \approx -\omega$ could easily be carried out in addition. In Fig. 12 the envelope of the oscillation executed is shown as a function of the difference $(\omega_\beta - \omega)$ between the resonant and the exciting frequency for different times t after the start of the excitation. As this time increases particles oscillating with opposite phase are close together in frequency leading to some cancellation in the integration to follow. For the resistive (in phase) term the oscillators with resonant frequency close to the exciting frequency gain large amplitudes.

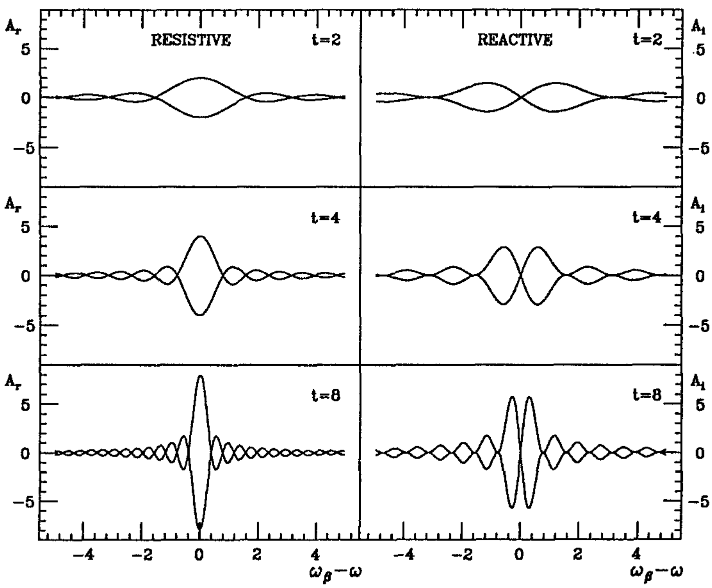


Figure 12: Time evolution of the beam response to a harmonic excitation

We calculate the centre-of-mass response by integrating the single-particle response Eq. (104) over the resonant frequency ω_β weighted with the distribution $f(\omega_\beta)$. Using again the symmetry property of the distribution $f(\omega_\beta) = f(-\omega_\beta)$ we can show that

$$\int_{-\infty}^{\infty} \frac{1-\cos((\omega_\beta-\omega)t)}{(\omega_\beta-\omega)} d\omega_\beta = -\int_{-\infty}^{\infty} \frac{1-\cos(\omega_\beta+\omega t)}{\omega_\beta+\omega} d\omega_\beta \tag{105}$$

and

$$\int_{-\infty}^{\infty} \frac{\sin((\omega_\beta - \omega)t)}{(\omega_\beta - \omega)} d\omega_\beta = \int_{-\infty}^{\infty} \frac{\sin(\omega_\beta + \omega t)}{\omega_\beta + \omega} d\omega_\beta \tag{106}$$

which gives for the velocity response

$$< \dot{y} >= \frac{\hat{G}\cos(\omega t)}{2\pi} \int_{-\infty}^{\infty} f(\omega_\beta) \frac{1 - \cos((\omega_\beta - \omega)t)}{\omega_\beta - \omega} d\omega_\beta + \frac{\hat{G}\sin(\omega t)}{2\pi} \int_{-\infty}^{\infty} f(\omega_\beta) \frac{\sin((\omega_\beta - \omega)t)}{\omega_\beta - \omega} d\omega_\beta. \tag{107}$$

We start with the out of phase (reactive) part which has an oscillatory term of the form $(\cos((\omega_\beta - \omega)t))$. As the time t increases this term will have opposite phase for smaller and smaller frequency differences in ω_β. The integration over ω_β will therefore vanish and we can replace the denominator of the reactive term by unity $(1 - \cos((\omega_\beta - \omega)t) \rightarrow 1$ except for the oscillators with $\omega_\beta \approx \omega$. This central part $\omega_\beta \approx \omega$ becomes more and more narrow as the time t increases and we can replace the integral over the reactive term by the principle value integral Eq. (95)

$$\lim_{t\to\infty} \int f(\omega_\beta) \frac{1 - \cos((\omega_\beta - \omega)t)}{\omega_\beta - \omega} d\omega_\beta = \lim_{\epsilon\to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) = PV \int f(\omega_\beta) \frac{1 - \cos((\omega_\beta - \omega)t)}{\omega_\beta - \omega} d\omega_\beta.. \tag{108}$$

The resistive (in phase) term of Eq. (107) contains an oscillatory term under the integral of the form

$$\frac{\sin((\omega_\beta - \omega)t)}{\omega_\beta - \omega} = t \frac{\sin((\omega_\beta - \omega)t)}{(\omega_\beta - \omega)t} = t\,\mathrm{sinc}((\omega_\beta - \omega)t). \tag{109}$$

For $\omega_\beta \neq \omega$ and large t the above expression oscillates with opposite phase for small changes in ω_β. The integration over this frequency will vanish as long as the distribution $f(\omega_\beta)$ is sufficiently smooth. For $\omega_\beta \approx \omega$ and very large t the function $\mathrm{sinc}((\omega_\beta - \omega)t)$ will be about unity and the above expression grows with increasing time t without limits. Furthermore the integral [9]

$$\int_{-\infty}^{\infty} \frac{\sin((\omega_\beta - \omega)t)}{(\omega_\beta - \omega)} d\omega_\beta = \pi \tag{110}$$

is independent of t. We can therefore replace Eq. (109) by the δ-function

$$\lim_{t\to\infty} \frac{\sin((\omega_\beta - \omega)t)}{(\omega_\beta - \omega)} = \pi\delta(\omega_\beta - \omega). \tag{111}$$

Collecting the results obtained for the reactive and resistive part of the beam response we get

$$\lim_{t\to\infty} < \dot{y} >= \frac{\hat{G}}{2\pi} \left[\sin(\omega t)\pi f(\omega) + \cos(\omega t) PV \int \frac{f(\omega_\beta)}{(\omega_\beta - \omega)} d\omega_\beta \right]. \tag{112}$$

This is the same result as that already obtained more quickly using complex notation. However, in this subsection we learned basically three things:

a) The result Eq. (96) is only correct if the excitation has lasted for a long time. How long this time has to be depends on the resolution with which the distribution $f(\omega_\beta)$ has to be considered. If this distribution does not change significantly over a frequency range of $\Delta\omega_\beta$ it is sufficient to excite for a time $t \gg 1/\Delta\omega_\beta$.

b) The sign of the residue can be determined from the initial conditions. Usually one excites a set of oscillators being initially at rest. However, it is in principle possible to have a set of particles oscillating initially with a particular distribution in amplitude and phase such that the exciting acceleration takes energy out of the beam. Since in the complex notation the initial conditions were not specified, both possibilities are contained in the beam response equation.

c) From Fig. 1 it is clear that a few oscillators, having resonant frequencies close to the exciting frequency $\omega_\beta \approx \omega$, attain large amplitudes. The energy absorbed by the beam from the exciter goes, therefore, into large oscillation amplitudes obtained by a small fraction of the oscillators.

5.5 Transverse impedance

It is well known that a beam can excite longitudinal modes in a cavity which then react onto the beam. It is also possible that the beam excites so-called deflecting modes which give a transverse force to the beam. A simple case is illustrated in Fig. 13 where a bunch is going through a cavity with a displacement y from the axis. This can excite a mode having a longitudinal electric field which increases with distance from the cavity axis. A quarter of an oscillation later this mode has a transverse magnetic field which can deflect particles. The transverse impedance is defined as the integrated deflecting field per unit dipole moment of the exciting current

$$Z_T(\omega) = -i\frac{\int_0^{2\pi R}[\vec{E}(\omega) + [\vec{\beta}c \times \vec{B}(\omega)]]_T ds}{Iy(\omega)}. \tag{113}$$

The right hand side is multiplied with i which indicates that the driving dipole moment Iy is out of phase with the deflecting field. The above expression is matched to a complex description of an oscillation with $e^{-i\omega t}$. If one uses the convention $e^{j\omega t}$ instead, i has to be replaced by $-j$. This impedance might become more clear if we relate the fields to the vertical velocity $\dot{y}$ rather than to the position y. With $y = \hat{y}e^{j\omega t}$ we have $\dot{y} = j\omega y$ and

$$Z_T(\omega) = -\omega\frac{\int_0^{2\pi R}[\vec{E}(\omega) + [\vec{\beta}c \times \vec{B}(\omega)]]_T ds}{I\dot{y}(\omega)}. \tag{114}$$

A real transverse impedance means that the transverse deflecting fields are in phase with the transverse velocity and transfer energy to or from the transverse motion of the beam.

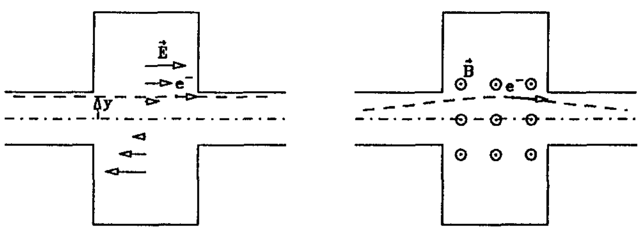


Figure 13: Transverse mode in a cavity during excitation and deflection

5.6 Transverse beam stability criterion

We calculated the centre-of-charge (coherent) response of the beam to an external transverse excitation with acceleration G

$$G = \frac{< e[\vec{E} + [\vec{\beta}c \times \vec{B}]] >}{m_0\gamma} \tag{115}$$

which gives for the fast wave

$$< y > = -\frac{\hat{G}e^{-i\omega t}}{2Q\omega_0}\int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega} d\omega_{\beta f}. \tag{116}$$

If the beam surroundings represent a transverse impedance Z_T,

$$Z_T(\omega) = -i\frac{\int_0^{2\pi R}[\vec{E}(\omega) + [\vec{\beta}c \times \vec{B}(\omega)]]_T ds}{Iy(\omega)}, \tag{117}$$

the oscillating beam will induce fields in it which will act back on the beam. We assume now that the acceleration G is not external but due to the self-fields induced

$$G = \frac{ieZ_T I < y >}{\gamma m_0 2\pi R}. \tag{118}$$

Substituting this in the expression for the beam response we get for the fast and the slow wave

$$1 = -\frac{icIZ_T}{4\pi Q m_0 c^2 \gamma} \int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega} d\omega_{\beta f} \text{ and } 1 = \frac{icIZ_T}{4\pi Q m_0 c^2 \gamma} \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s}. \tag{119}$$

This gives a condition for which the beam is just at the limit of stability [10, 11]. By replacing the external excitation with the self induced one we assume that an oscillation, once started, is just kept going by the self-forces. To make it more applicable we introduce some normalization to separate the term which depends on the different beam parameters from the one which is just given by the form of the distribution. We introduce the half width at half height S of the distribution $f(\omega_\beta)$ and normalize the two frequencies ω and ω_β with it

$$\xi_f = \frac{\omega_{\beta f}}{S} \ \xi_s = \frac{\omega_{\beta s}}{S},\ \xi = \frac{\omega}{S},\ f(\xi_f) = Sf(\omega_{\beta f}),\ f(\xi_s) = Sf(\omega_{\beta s}) \tag{120}$$

and get

$$1 = \frac{iecIZ_T}{4\pi EQS} \int \frac{f(\xi_f)}{\xi - \xi_f} d\xi_f \text{ and } 1 = -\frac{iecIZ_T}{4\pi EQS} \int \frac{f(\xi_s)}{\xi - \xi_s} d\xi_s. \tag{121}$$

This expression allows us to find, for given beam parameters, for each value of the driving frequency ω (or its normalized value ξ), the maximum impedance Z_T which still does not lead to an instability. In this expression the integral and the impedance are complex numbers. It is therefore convenient to visualize the stability criterion by mapping the impedance to the frequency ω or ξ_1. As long as this frequency has no imaginary part there is no growing instability since we expressed the oscillation as $e^{-i\omega t}$. We write the above condition slightly differently

$$\frac{ecIZ_T}{4\pi EQS} = V_T + iU_T = \frac{1}{i\int \frac{f(\xi_f)}{\xi - \xi_f} d\xi_f} \text{ and } \frac{ecIZ_T}{4\pi EQS} = V_T + iU_T = -\frac{1}{i\int \frac{f(\xi_s)}{\xi - \xi_s} d\xi_s}. \tag{122}$$

Plotting the above equality for a real frequency ξ_1 gives the stability diagram. We give in Fig. 14 the example of a Gaussian distribution. The diagram consists of two curves; at the right, one for the slow wave with positive values of the impedance; at the left, one for the fast wave with negative values for the resistive impedance. These curves represent a situation which is at the limit of stability. A slightly larger impedance than the one corresponding to this limit will lead to an instability. For the slow wave this means that values for the complex impedance Z_T giving a reduced impedance $U_T + iV_T$ lying on the right of the stability diagram lead to instability while for corresponding values lying on the left we still have stability. For the fast wave only negative impedances lead to instabilities if the resulting reduced impedance $U_T + iV_T$ lies on the left of the fast wave stability curve. To summarize the situation for both waves, we have stability as long as the reduced impedance $U_T + iV_T$ is inside the stability diagram bounded by the fast and the slow wave stability limit curves.

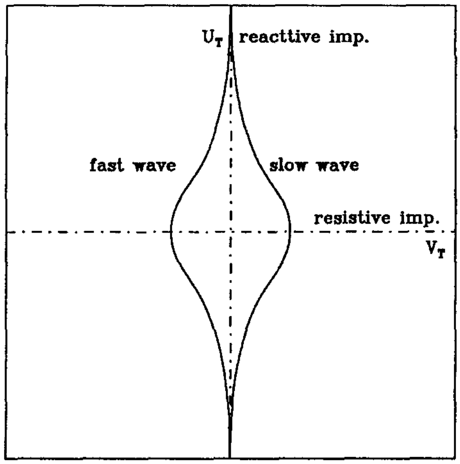


Figure 14: Stability diagram

5.7 Landau damping for bunched beams and for longitudinal instabilities

Without going into any details concerning the Landau damping for bunched beams and for longitudinal oscillations we should make a few remarks on the limitation of the treatment presented here. The Landau damping against transverse instabilities in an unbunched beam which has a betatron frequency spread due to the energy spread combined with sextupoles Eq. (86) is a special case. The betatron oscillation we excite will not influence the frequency spread in the beam. In other words, the frequency ω_β of a particle is given by parameters like Δp, η and Q' which are not influenced by the excitation of betatron oscillations. This situation is different and more complicated in cases where the betatron frequency spread is determined by octupole fields which give a dependence of the betatron frequency on amplitude. This is always the case for a bunched beam but it can also be the dominant effect in unbunched beams. Exciting a betatron oscillation will, at the same time, influence the betatron frequency distribution. Going through this calculation one finds that the integrals, Eq. (95), determining the beam response and the stability diagram do not contain the distribution $f(\omega_\beta)$ but rather its derivative [10-12] The same situation is present for longitudinal stability for unbunched as well as for bunched beams [12-16]

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References

[1] K.W. Robinson, Stability of beam in radiofrequency system, Cambridge Electron Accel. CEAL-1010 (1964).

[2] F. Sacherer, A longitudinal stability criterion for bunched beams, Proc. of the 1977 Particle Accel. Conf., IEEE Trans. on Nucl. Sci. NS 20-3 (1973) 825.

[3] J.L. Laclare, Bunched beam coherent instabilities, CAS CERN Accelerator School, Advanced Accelerator Physics, 1985; ed. S. Turner. CERN 87-03, p. 264.

[4] F. Pedersen, CERN, Private communication.

[5] S. Krinsky, Saturation of a longitudinal instability due to nonlinearity of the wake field, Proc. of the 1985 Particle Accel. Conf., IEEE Trans. on Nucl. Sci. NS 32-5 (1985) 2320.

[6] A. Hofmann, Physics of beam instabilities, Proc of a Topical Course 'Frontiers of Particle Beams' held by the Joint US-CERN School on Particle Accel. at South Padre Island, Texas, Oct. 1986, ed. M. Month and S. Turner, Lecture Notes in Physics 296. Springer (1988) 99.

[7] A. Hofmann, Landau damping, CAS CERN Accelerator School, Advanced Accelerator Physics, 1987; ed. S. Turner. CERN 89-01, p. 40.

[8] S. van der Meer, A different formulation of the longitudinal and transverse beam response, CERN/PS/AA/80-4 (1980).

[9] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals Series and Products, Academic Press 1980, p. 406.

[10] L.J. Laslett, V.K. Neil, A.M. Sessler, Transverse resistive instabilities of intense coasting beams in particle accelerators, Rev. Sci. Instr. 36 No. 4 (1965) 436.

[11] K. Hübner and V. Vaccaro, Dispersion relations and stability of coasting particle beams, CERN/ISR-TH-RF/69-23 (1969).

[12] V.K. Neil, A.M. Sessler, Longitudinal resistive instabilities of intense coasting beams in particle accelerators, Rev. Sci. Instr. 36 No. 4 (1965) 429.

[13] H.G. Hereward, Landau damping by non-linearities, CERN/MPS/DL 69-11 (1969).

[14] D. Boussard, Schottky noise and beam transfer function diagnostics, CAS CERN Accelerator School, Advanced Accelerator Physics, 1985; ed. S. Turner. CERN 87-03, p. 416.

[15] J. Gareyte, Landau damping of the longitudinal quadrupole mode in SPEAR II, SLAC SPEAR-207 (1977).

[16] R. Boni, S. Guiducci, M. Serio, F. Tazzioli, F.H. Wang, Investigations on beam longitudinal transfer function and coupling impedance in ADONE, Frascati LNF-Div. Report, RM-23,1981.